

Clique minors in double-critical graphs

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Abstract

A connected t -chromatic graph G is *double-critical* if $G \setminus \{u, v\}$ is $(t - 2)$ -colorable for each edge $uv \in E(G)$. A long standing conjecture of Erdős and Lovász that the complete graphs are the only double-critical t -chromatic graphs remains open for all $t \geq 6$. Given the difficulty in settling Erdős and Lovász's conjecture and motivated by the well-known Hadwiger's conjecture, Kawarabayashi, Pedersen and Toft proposed a weaker conjecture that every double-critical t -chromatic graph contains a K_t minor and verified their conjecture for $t \leq 7$. A computer-assisted proof of their conjecture for $t = 8$ was recently announced by Albar and Gonçalves. In this paper we give a much shorter and computer-free proof of their conjecture for $t \leq 8$ and prove the next step by showing that every double-critical t -chromatic graph contains a K_9 minor for all $t \geq 9$.

1 Introduction

All graphs in this paper are finite and simple. For a graph G we use $|G|$, $e(G)$, $\delta(G)$ to denote the number of vertices, number of edges and minimum degree of G , respectively. The degree of a vertex v in a graph is denoted by $d_G(v)$ or simply $d(v)$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$ and $G \setminus S = G[V(G) - S]$. For a subgraph H of G , $G \setminus H = G[V(G) - V(H)]$. If G is a graph and K is a subgraph of G , then by $N(K)$ we denote the set of vertices of $V(G) - V(K)$ that are adjacent to a vertex of K . If $V(K) = \{x\}$, then we use $N(x)$ to denote $N(K)$. By abusing notation we will also denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x] = N(x) \cup \{x\}$, and similarly will use the same symbol for the graph induced by that set. If u, v are distinct nonadjacent vertices of a graph G , then by $G + uv$ we denote the graph obtained from G by adding an edge with ends u and v . If u, v are adjacent or equal, then we define $G + uv$ to be G .

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. We write $G > H$ if H is a minor G . In those circumstances we also say that G has an H *minor*. A connected graph G is called *double-critical* if for any edge $uv \in E(G)$, we have $\chi(G \setminus \{u, v\}) = \chi(G) - 2$. The following long standing *Double-Critical Graph Conjecture* is due to Erdős and Lovász [2].

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Conjecture 1.1 Double-Critical Graph Conjecture: For every integer $t \geq 1$, the only double-critical t -chromatic graph is K_t .

Conjecture 1.1 is a special case of the so-called Erdős-Lovász Tihany Conjecture [2]. It is trivially true for $t \leq 3$ and reasonably easy for $t = 4$. Mozhan [7] and Stiebitz [9] independently proved Conjecture 1.1 for $t = 5$.

Theorem 1.2 The only double-critical 5-chromatic graph is K_5 .

Conjecture 1.1 remains open for all $t \geq 6$. Given the difficulty in settling Conjecture 1.1 and motivated by the well-known Hadwiger's conjecture, Kawarabayashi, Pedersen and Toft proposed a weaker conjecture.

Conjecture 1.3 For every integer $t \geq 1$, every double-critical t -chromatic graph contains a K_t minor.

Conjecture 1.3 is a weaker version of Hadwiger's conjecture [3], and is true for $t \leq 5$ by Theorem 1.2. In the same paper [5], Kawarabayashi, Pedersen and Toft verified their conjecture for $t = 6, 7$.

Theorem 1.4 For every integer $t \leq 7$, every double-critical t -chromatic graph contains a K_t minor.

Recently, Albar and Gonçalves [1] announced a proof for the case $t = 8$.

Theorem 1.5 Every double-critical 8-chromatic graph has a K_8 minor.

Our main result is the following next step.

Theorem 1.6 For integers k, t with $1 \leq k \leq 9$ and $t \geq k$, every double-critical t -chromatic graph contains a K_k minor.

We actually prove a much stronger result, the following.

Theorem 1.7 For $6 \leq k \leq 9$, let G be a $(k - 3)$ -connected graph with $k + 1 \leq \delta(G) \leq 2k - 5$. If every edge of G is contained in at least $k - 2$ triangles and for any minimal separating set S of G and any $x \in S$, $G[S \setminus x]$ is not a clique, then $G > K_k$.

Theorem 1.6 follows directly from Proposition 2.1 (see below) and Theorem 1.7. Our proof of Theorem 1.7 uses the main idea in the proof of Theorem 1.10. We want to point out that the proof of Theorem 1.4 for $k = 7$ is about ten pages long and the proof of Theorem 1.5 is computer-assisted and not simple. Our proof of Theorem 1.6 is much shorter and computer-free for $k \leq 8$. For $k = 9$, our proof is also computer-assisted as it applies a lemma from [8] (see Lemma 1.14 below). Notice that a computer-assisted proof of Theorem 1.7 for all $k \leq 8$ (and hence computer-assisted proofs of Theorem 1.4 and Theorem 1.5) follows directly from Theorem 1.7 for $k = 9$. (To see that, let G and $k \leq 8$ be as in Theorem 1.7, and let H be obtained from G by adding $9 - k$ vertices, each joined

to every other vertex of the graph. Then H is 6-connected and satisfies all the other conditions as stated in Theorem 1.7. Thus $H > K_9$ and so $G > K_k$.) Conjecture 1.3 remains open for all $t \geq 10$. It seems hard to generalize Theorem 1.6.

We need some known results to prove our main results. Before doing so, we need to define (H, k) -cockade. For a graph H and an integer k , let us define an (H, k) -cockade recursively as follows. Any graph isomorphic to H is an (H, k) -cockade. Now let G_1, G_2 be (H, k) -cockades and let G be obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2 . Then the graph G is also an (H, k) -cockade, and every (H, k) -cockade can be constructed this way. We are now ready to state some known results. The following theorem is a result of Mader [6].

Theorem 1.8 For every integer $p = 1, 2, \dots, 7$, a graph on $n \geq p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.

Jørgensen [4] and later the second author and Thomas [8] generalized Theorem 1.8 to $p = 8$ and $p = 9$, as follows.

Theorem 1.9 Every graph on $n \geq 8$ vertices with at least $6n - 20$ edges either contains a K_8 -minor or is isomorphic to a $(K_{2,2,2,2,2}, 5)$ -cockade.

Theorem 1.10 Every graph on $n \geq 9$ vertices with at least $7n - 27$ edges either contains a K_9 -minor, or is isomorphic to $K_{2,2,2,3,3}$, or is isomorphic to a $(K_{1,2,2,2,2,2}, 6)$ -cockade.

It seems hard to generalize Theorem 1.8 for all values of p . In 2003, Seymour and Thomas [8] proposed the following conjecture.

Conjecture 1.11 For every $p \geq 1$ there exists a constant $N = N(p)$ such that every $(p-2)$ -connected graph on $n \geq N$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.

By Theorem 1.10, Conjecture 1.11 is true for $p \leq 9$.

In our proof of Theorem 1.7, we need to examine graphs G such that $k+1 \leq |V(G)| \leq 2k-5$, $\delta(G) \geq k-2$ and $G \not\geq K_k \cup K_1$. So the following lemmas will be needed. Lemma 1.12 is a result of Jørgensen [4] and Lemma 1.14 is due to the second author and Thomas [8]. Notice that the proof of Lemma 1.14 is computer-assisted. One can easily see that Lemma 1.12 implies Lemma 1.13. To see that, let G and t be as in Lemma 1.13, and apply Lemma 1.12 to the graph obtained from G by adding $6-t$ vertices, each adjacent to every other vertex of the graph.

Lemma 1.12 Let G be a graph with $n \leq 11$ vertices and $\delta(G) \geq 6$ such that for every vertex x in G , $G - x$ is not contractible to K_6 . Then G is one of the graphs $K_{2,2,2,2}, K_{3,3,3}$ or the complement of the Petersen graph.

Lemma 1.13 For $1 \leq t \leq 5$, let G be a graph with $n \leq 2t - 1$ vertices and $\delta(G) \geq t$. Then $G > K_t \cup K_1$.

Lemma 1.14 Let n be an integer satisfying $9 \leq n \leq 13$ and let G be a graph on n vertices with $\delta(G) \geq 7$. Then either $G > K_7 \cup K_1$, or G satisfies the following

- (A) either G is isomorphic to $K_{1,2,2,2,2}$, or G has four distinct vertices a_1, b_1, a_2, b_2 such that $a_1a_2, b_1b_2 \notin E(G)$ and for $i = 1, 2$ the vertex a_i is adjacent to b_i , the vertices a_i, b_i have at most four common neighbors, and $G + a_1a_2 + b_1b_2 > K_8$,
- (B) for any two sets $A, B \subseteq V(G)$ of cardinality at least five such that neither is complete and $A \cup B$ includes all vertices of G of degree at most $|G| - 2$, either
 - (B1) there exist $a \in A$ and $b \in B$ such that $G' > K_8$, where G' is obtained from G by adding all edges aa' and bb' for $a' \in A - \{a\}$ and $b' \in B - \{b\}$, or
 - (B2) there exist $a \in A - B$ and $b \in B - A$ such that $ab \in E(G)$ and the vertices a and b have at most five common neighbors in G , or
 - (B3) one of A and B contains the other and $G + ab > K_7 \cup K_1$ for all distinct nonadjacent vertices $a, b \in A \cap B$.

2 Basic properties of non-complete double-critical graphs

We begin with basic properties of non-complete double-critical k -chromatic graphs established in [5]. We only list those that will be used in our proofs.

Proposition 2.1 If G is a non-complete double-critical k -chromatic graph, then the following hold:

- (a) $\delta(G) \geq k + 1$.
- (b) Every edge $xy \in E(G)$ belongs to at least $k - 2$ triangles.
- (c) G is 6-connected and no minimal separating set of G can be partitioned into two sets A and B such that $G[A]$ and $G[B]$ are edge-empty and complete, respectively.

It seems hard to use the main idea in the proof of Proposition 2.1 (c) to prove that any non-complete double-critical k -chromatic graph is 7-connected. But we can say a bit more about minimal separating sets of size 6 in such graphs. Two proper vertex-colorings c_1 and c_2 of a graph G are *equivalent* if, for all $x, y \in V(G)$, $c_1(x) = c_1(y)$ iff $c_2(x) = c_2(y)$. For any $A \subseteq V(G)$. We say that two vertex-colorings c_1 and c_2 of G are *equivalent on A* if the restrictions $c_{1|A}$ and $c_{2|A}$ to A are equivalent on the subgraph $G[A]$. Let S be a separating set of G , and let G_1, G_2 be connected subgraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = G[S]$. If c_1 is a k -coloring of G_1 and c_2 is a k -coloring of G_2 such that c_1 and c_2 are equivalent on S , then it is clear that c_1 and c_2 can be combined to a k -coloring of G by a suitable permutation of the color classes of, say c_1 .

Lemma 2.2 Suppose G is a non-complete double-critical k -chromatic graph. If S is a minimal separating set of G with $|S| = 6$, then either $G[S] \subseteq K_{3,3}$ or $G[S] \subseteq K_{2,2,2}$.

Proof. Suppose G is a non-complete double-critical k -chromatic graph. By Proposition 2.1 (c), G is 6-connected. Let $S = \{v_1, \dots, v_6\} \subset V(G)$ be a minimal separating set of G such that neither $G[S] \subseteq K_{3,3}$ nor $G[S] \subseteq K_{2,2,2}$. Let H be a component of $G \setminus S$, and let $G_1 = G[V(H) \cup S]$ and $G_2 = G \setminus V(H)$. Then $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = S$. Since $k \geq 6$ by Proposition 2.1 (c), we have $\delta(G) \geq 7$ by Proposition 2.1 (a). In particular, since $|S| = 6$, there must exist at least one edge in each of $G_1 \setminus S$ and $G_2 \setminus S$. It follows then that both G_1 and G_2 are $(k-2)$ -colorable. Let c_1, c_2 be $(k-2)$ -colorings of G_1 and G_2 , respectively. For $i = 1, 2$, define $|c_i(S)|$ to be the number of distinct colors assigned to the vertices of S by c_i . We may assume that $|c_1(S)| \geq |c_2(S)|$. Clearly c_1 and c_2 are not equivalent on S , otherwise c_1 and c_2 , after a suitable permutation of the colors of c_2 , can be combined to a $(k-2)$ -coloring of G , a contradiction. Thus $|c_2(S)| \leq 5$. By Proposition 2.1 (c), $\alpha(G[S]) \leq 4$ and so neither c_1 nor c_2 applies the same color to more than four vertices of S . Utilizing a new color, say α , we next redefine the colorings c_1 and c_2 so that c_1 and c_2 are $(k-1)$ -colorings of G_1 and G_2 , respectively, and are equivalent on S . This yields a contradiction, as c_1 and c_2 , after a suitable permutation of the colors of c_2 , can be combined to a $(k-1)$ -coloring of G .

Suppose that one of the colorings c_1 and c_2 , say c_1 , assigns the same color to four vertices of S , say $c_1(v_3) = c_1(v_4) = c_1(v_5) = c_1(v_6)$. Then $\{v_3, v_4, v_5, v_6\}$ is an independent set in G . Since $G[S] \not\subseteq K_{2,2,2}$, we have $c_2(v_1) \neq c_2(v_2)$. Now redefining $c_2(v_3) = c_2(v_4) = c_2(v_5) = c_2(v_6) = \alpha$ and $c_1(v_1) = \alpha$ will, after a suitable permutation of the colors of c_2 , make c_1 and c_2 equivalent on S using $k-1$ colors. Thus neither c_1 nor c_2 assigns the same color to four distinct vertices of S .

Next suppose that one of the colorings c_1 and c_2 , say c_1 , assigns the same color to three vertices of S , say $c_1(v_4) = c_1(v_5) = c_1(v_6)$. Then $\{v_4, v_5, v_6\}$ is an independent set in G . Since $G[S] \not\subseteq K_{3,3}$, we have $|c_2(\{v_1, v_2, v_3\})| \geq 2$. If $|c_2(\{v_1, v_2, v_3\})| = 2$, we may assume that $c_2(v_2) = c_2(v_3)$. Then $\{v_2, v_3\}$ is an independent set. Then redefining $c_2(v_4) = c_2(v_5) = c_2(v_6) = \alpha$ and $c_1(v_2) = c_1(v_3) = \alpha$ will, after a suitable permutation of the colors of c_2 , make c_1 and c_2 equivalent on S using $k-1$ colors. Thus $|c_2(\{v_1, v_2, v_3\})| = 3$ and so c_2 assigns distinct colors to each of v_1, v_2, v_3 . We redefine $c_2(v_4) = c_2(v_5) = c_2(v_6) = \alpha$. Clearly c_1 and c_2 are equivalent on S if c_1 assigns distinct colors to each of v_1, v_2, v_3 . Thus $|c_1(\{v_1, v_2, v_3\})| \leq 2$. Since $G[S] \not\subseteq K_{3,3}$, we have $|c_1(\{v_1, v_2, v_3\})| = 2$. We may assume that $c_1(v_2) = c_1(v_3)$. Now redefining $c_1(v_3) = \alpha$ yields that, after a suitable permutation of the colors of c_2 , c_1 and c_2 are equivalent on S . This proves that neither c_1 nor c_2 assigns the same color to three distinct vertices of S . Thus $5 \geq |c_i(S)| \geq 3$ ($i = 1, 2$). Since $G[S] \not\subseteq K_{2,2,2}$, we have $|c_2(S)| \geq 4$. Thus $5 \geq |c_2(S)| \geq 4$.

Suppose that $|c_2(S)| = 5$. Then $|c_1(S)| = 5$ or $|c_1(S)| = 6$. We can make c_1 and c_2 equivalent on S by assigning color α to one of the two vertices that are colored the same color by c_1 (if $|c_1(S)| = 5$) and c_2 . Thus $|c_2(S)| = 4$. Since neither c_1 nor c_2 assigns the same color to more

than two distinct vertices of S , we may assume that $c_2(v_3) = c_2(v_4)$ and $c_2(v_5) = c_2(v_6)$. Then $v_3v_4, v_5v_6 \notin E(G)$. Since $G[S] \not\subseteq K_{2,2,2}$, we have $v_1v_2 \in E(G)$. Thus $c_1(v_1) \neq c_1(v_2)$. We may assume that $c_1(v_3) \neq c_1(v_4)$ as c_1 and c_2 are not equivalent on S . We redefine $c_2(v_3) = \alpha$. Then the new c_2 is a $(k-1)$ -coloring of G_2 with $|c_2(S)| = 5$. Clearly c_1 and the new c_2 are not equivalent on S . If $|c_1(S)| \geq 5$, then $c_1(v_5) \neq c_1(v_6)$ and we may assume v_5 is one of the two vertices that are colored the same color by c_1 if $|c_1(S)| = 5$. Now redefining $c_1(v_5) = c_1(v_6) = \alpha$ will make c_1 and c_2 equivalent on S . Thus $|c_1(S)| = 4$. Suppose $c_1(v_5) = c_1(v_6)$. Since $v_1v_2 \in E(G)$, we may assume that $c_1(v_1) = c_1(v_3)$. Now redefining $c_1(v_3) = \alpha$ will make c_1 and c_2 equivalent on S . Thus $c_1(v_5) \neq c_1(v_6)$. Let A and B be the two color classes of c_1 on S with $|A| = |B| = 2$. Suppose $v_1 \in A$ and $v_2 \in B$. We may assume that $v_3 \in A$. Then $v_4 \notin B$ because $G[S] \not\subseteq K_{2,2,2}$ and $v_1v_3 \notin E(G)$. We may assume that $v_5 \in B$. Redefining $c_1(v_5) = c_1(v_6) = \alpha$ and $c_2(v_1) = \alpha$ will make c_1 and c_2 equivalent on S . Thus $v_1, v_2 \notin A \cup B$. By symmetry, we may assume $B = \{v_3, v_5\}$. We may further assume that $v_6 \in A$. Now redefining $c_1(v_5) = c_1(v_6) = \alpha$ will make c_1 and c_2 equivalent on S .

This completes the proof of Lemma 2.2. ■

3 Proofs of Theorem 1.7 and Theorem 1.6

In this section we first prove Theorem 1.7.

Proof. Let G be a graph as in the statement with n vertices. By assumption, we have

- (1) $k+1 \leq \delta(G) \leq 2k-5$ and $\delta(N(x)) \geq k-2$ for any x in G ; and
- (2) G is $(k-3)$ -connected and for any minimal separating set S of G and any $x \in S$, $G[S \setminus x]$ is not a complete subgraph.

We first show that the statement is true for $k = 6$. Assume $k = 6$. Then G is 3-connected with $\delta(G) = 7$. The statement is trivially true if G is complete, so we may assume G is not complete. Let $x \in V(G)$ be a vertex of degree seven. By (1), $\delta(N(x)) \geq 4$, and so $e(N(x)) \geq 14$. If $e(N(x)) \geq 16$, then by Theorem 1.8, $N(x) > K_5$ and so $G > N[x] > K_6$. If $e(N(x)) = 15$, then let K be a component of $G - N[x]$. By (2), $|N(K)| \geq 3$ and $N(K)$ is not complete. Let $x, y \in N(K)$ be non-adjacent in $N(x)$ and let P be an (x, y) -path with interior vertices in K . We see that $G > K_6$ by contracting all but one of the edges of P . So we may assume that $e(N(x)) = 14$, and so $N(x)$ is 4-regular and $\overline{N(x)}$ is 2-regular. Thus $\overline{N(x)}$ is then either isomorphic to C_7 or to $C_4 \cup C_3$, and in both cases it is easy to see that $N(x) > K_5$ and thus $G > K_6$, as desired. Hence we may assume $7 \leq k \leq 9$.

Suppose for a contradiction that $G \not\prec K_k$. We next prove the following.

(3) Let $x \in V(G)$ be such that $k+1 \leq d(x) \leq 2k-5$. Then there is no component K of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$, where M is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$.

Proof. Suppose such a component K exists. Among all vertices x with $k+1 \leq d(x) \leq 2k-5$ for which such a component exists, choose x to be of minimal degree. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M - N(K) \neq \emptyset$, and let $y \in M - N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y) < d(x)$. Let J be the component of $G - N[y]$ containing K . Since $d(y) < d(x)$ the choice of x implies that $N(x) - N[y] \not\subseteq V(J)$. Let $H = N(x) - N[y] - N(K)$. We have $d_G(z) \geq d_G(y)$ for all $z \in V(H)$ by the choice of y . Let $t = |V(H)|$. Then $t \geq 2$, for otherwise the vertex y and component H contradict the choice of x . On the other hand $t \leq d(x) - d(y) \leq (2k-5) - (k+1) = k-6 \leq 3$ and so $k \geq 8$. Notice that $t = 2$ when $k = 8$. From (1) applied to y we deduce that $N(y) \cap N(x)$ has minimum degree at least $k-3$. Let L be the subgraph of G induced by $(N[y] \cap N(x)) \cup V(H)$. Then the edge-set of L consists of edges of $N(x) \cap N(y)$, edges incident with y , and edges incident with $V(H)$. Clearly, $e(L \setminus H, H) = \sum_{z \in V(H)} (d(z) - 1) - 2e(H) \geq t(d(y) - 1) - 2e(H)$. Thus

$$\begin{aligned}
e(L) &\geq \frac{(k-3)(d(y)-1)}{2} + d(y) - 1 + e(L \setminus H, H) + e(H) \\
&\geq \frac{(k-3)(d(y)-1)}{2} + d(y) - 1 + t(d(y) - 1) - e(H) \\
&\geq \frac{(k-3)(d(y)-1)}{2} + d(y) - 1 + t(d(y) - 1) - \frac{1}{2}t(t-1) \\
&\geq \begin{cases} 5(d(y)+2) + \frac{d(y)}{2} - \frac{33}{2} & \text{if } k=8 \\ 6(d(y)+t) + (t-2)d(y) - 4 - 7t - \frac{1}{2}t(t-1) & \text{if } k=9 \end{cases} \\
&\geq (k-3)|V(L)| - \binom{k-2}{2} + 1,
\end{aligned}$$

because $d(y) \geq k+1$ and $2 \leq t \leq k-6$. If $k = 9$, since $12 \leq |V(L)| \leq 13$ the graph L is not a $(K_{2,2,2,2,2}, 5)$ -cockade. By Theorem 1.8 and Theorem 1.9, $N(x) > L > K_{k-1}$. Thus $G > N[x] > K_k$, a contradiction. This proves that $M \subseteq N(K)$.

If $N(x) > K_{k-2} \cup K_1$, then $N(x)$ has a vertex y such that $N(x) - y > K_{k-2}$. If $y \notin M$, then $N(x) > K_{k-1}$. Otherwise, by contracting the connected set $V(K) \cup \{y\}$ we can contract K_{k-1} onto $N(x)$. Thus in either case $G > K_k$, a contradiction. Thus $N(x) \not> K_{k-2} \cup K_1$. If $k \leq 8$, by Lemma 1.12 and Lemma 1.13, we have $k = 8$ and $N(x)$ is either $K_{3,3,3}$ or \overline{P} , where P is the complement of the Petersen graph. If $N(x) = \overline{P}$. It can be easily checked that $\overline{P} + xy > K_7$ for any $xy \in E(P)$. By (2), $|N(K)| \geq 5$ and $N(K)$ is not complete. Let $x, y \in N(K)$ be non-adjacent in $N(x)$ and let Q be an (x, y) -path with interior vertices in K . We see that $G > K_8$ by contracting all but one of the edges of Q , a contradiction. Thus $N(x) = K_{3,3,3}$. Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be the vertex sets of two disjoint triangles of $\overline{N(x)}$. Suppose $G - N[x]$ is 2-connected or has at most two vertices. Clearly, the vertices a_i, b_i ($i=1,2$) have at least two common neighbors

in $G - N[x]$. Let u_1, u_2 (resp. w_1, w_2) be two distinct common neighbors of a_1 and b_1 (resp. a_2 and b_2) in $G - N[x]$. By Menger's Theorem, $G - N[x]$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$ and so $G > N[x] + a_1a_2 + b_1b_2 > K_8$, a contradiction. Thus $G - N[x]$ has at least three vertices and is not 2-connected. If $G - N[x]$ is disconnected, let $H_1 = K$ and H_2 be another connected component of $G - N[x]$. If $G - N[x]$ has a cut-vertex, say w , let H_1 be a connected component of $G - N[x] - w$ and let $H_2 = G - N[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N[x]$ such that $M \subseteq N(H_1) \cup N(H_2)$ (because we have shown that $M \subseteq N(K)$). By (2), $N(H_i)$ is not complete and $|N(H_i)| \geq 4$. By the pigeonhole principle, we may assume that $\overline{N(H_2)}$ contains a matching of size at least two. Since $N(x) = K_{3,3,3}$, we may assume that $a_1a_2 \in \overline{N(H_1)}$ and $b_1b_2 \in \overline{N(H_2)}$. By contracting H_1 onto a_1 and H_2 onto b_1 we see that $G > N[x] + a_1a_2 + b_1b_2 > K_8$, a contradiction. This proves that $k = 9$ and so by Lemma 1.14, we may assume that $N(x)$ satisfies properties (A) and (B).

Since $d(x) \geq 10$, $N(x) \neq K_{1,2,2,2,2}$. If $G - N[x]$ is 2-connected or has at most two vertices, then by property (A) and (2), the set $N(x)$ has four distinct vertices a_1, b_1, a_2, b_2 such that $a_1a_2, b_1b_2 \notin E(G)$, $N(x) + a_1a_2 + b_1b_2 > K_8$ and for $i = 1, 2$ the vertex a_i is adjacent to b_i , and the vertices a_i, b_i have at least two common neighbors in $G - N[x]$. Let u_1, u_2 (resp. w_1, w_2) be two distinct common neighbors of a_1 and b_1 (resp. a_2 and b_2) in $G - N[x]$. By Menger's Theorem, $G - N[x]$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$ and so $G > N[x] + a_1a_2 + b_1b_2 > K_9$, a contradiction.

Thus $G - N[x]$ has at least three vertices and is not 2-connected. If $G - N[x]$ is disconnected, let $H_1 = K$ and H_2 be another connected component of $G - N[x]$. If $G - N[x]$ has a cut-vertex, say w , let H_1 be a connected component of $G - N[x] - w$ and let $H_2 = G - N[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N[x]$ such that $M \subseteq N(H_1) \cup N(H_2)$ (because we have shown that $M \subseteq N(K)$). For $i = 1, 2$ let $A_i = N(H_i) \cap N(x)$. By (2), A_i is not complete and $|A_i| \geq 5$ for $i = 1, 2$. By property (B), A_1 and A_2 satisfy properties (B1), (B2) or (B3).

Suppose first that A_1 and A_2 satisfy property (B1). Then there exist $a_i \in A_i$ such that $N(x) + \{a_1a : a \in A_1 - \{a_1\}\} + \{a_2a : a \in A_2 - \{a_2\}\} > K_8$. By contracting the connected sets $V(H_1) \cup \{a_1\}$ and $V(H_2) \cup \{a_2\}$ to single vertices, we see that $G > K_9$, a contradiction. Suppose next that A_1 and A_2 satisfy property (B2). Then there exist $a_1 \in A_1 - A_2$ and $a_2 \in A_2 - A_1$ such that $a_1a_2 \in E(G)$ and the vertices a_1 and a_2 have at most five common neighbors in $N(x)$. Thus $a_1, a_2 \in M$ by (1), and by another application of (1) there exists a common neighbor $u \in V(G) - N[x]$ of a_1 and a_2 . But $a_1 \notin A_2$ and $a_2 \notin A_1$, and hence $u \notin V(H_1) \cup V(H_2)$. Thus $G - N[x]$ is disconnected and $H_1 = K$. But then $a_2 \in M \subseteq N(K) = N(H_1)$, a contradiction. Thus we may assume that A_1 and A_2 satisfy (B3), and hence $A_i \subseteq A_{3-i}$ for some $i \in \{1, 2\}$. As $M \subseteq A_1 \cup A_2$, we have $M \subseteq N(H_{3-i})$. Since A_i is not complete, let $a, b \in A_i$ be distinct and not adjacent. By property (B3), $N(x) + ab > K_7 \cup K_1$. Let P be an a - b path with interior in H_i . By contracting all but one of the edges of the path P and by contracting H_{3-i} similarly as above, we see that $G > K_9$, a contradiction. ■

(4) $G - N[x]$ is disconnected for every vertex $x \in V(G)$ of degree at most $2k - 5$.

Proof. If $G - N[x]$ is not null, then it is disconnected by (3). Thus we may assume that x is adjacent to every other vertex of G . Let $H = G - x$. Then $|H| = d(x)$ and $\delta(H) \geq k$. Thus $e(H) \geq \frac{k d(x)}{2} > (k - 3) d(x) - \binom{k-2}{2} + 1$ because $d(x) \leq 2k - 5$. By Theorem 1.8 and Theorem 1.9, $G - x$ has a K_{k-1} minor and so the graph G has a K_k minor, a contradiction. ■

(5) Let $x \in V(G)$ be such that $k + 1 \leq d(x) \leq 2k - 5$. Then there is no component K of $G - N[x]$ such that $d_G(y) \geq 2k - 4$ for every vertex $y \in V(K)$.

Proof. Assume that such a component K exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let d_1 be the maximum number of edges that can be added to G_2 by contracting edges of G with at least one end in G_1 . More precisely, let d_1 be the largest integer so that G_1 contains disjoint sets of vertices V_1, V_2, \dots, V_p so that $G_1[V_j]$ is connected, $|N(K) \cap V_j| = 1$ for $1 \leq j \leq p = |N(K)|$, and so that the graph obtained from G_1 by contracting V_1, V_2, \dots, V_p and deleting $V(G) - (\bigcup_j V_j)$ has $e(N(K)) + d_1$ edges. Let G'_2 be a graph with $V(G'_2) = V(G_2)$ and $e(G'_2) = e(G_2) + d_1$ edges obtained from G by contracting edges in G_1 . By (1), $|G'_2| \geq k + 2$. If $e(G'_2) \geq (k - 2)|G'_2| - \binom{k-1}{2} + 2$, then by Theorem 1.8 and Theorem 1.9, $G > G'_2 > K_k$, a contradiction. Thus

$$e(G_2) = e(G'_2) - d_1 \leq (k - 2)|G_2| - \binom{k-1}{2} + 1 - d_1 = (k - 2)|N(K)| + (k - 2)|K| - \binom{k-1}{2} + 1 - d_1.$$

By contracting the edge xz , where $z \in N(K)$ has minimum degree d in $N(K)$, we see that $d_1 \geq |N(K)| - d - 1$ and hence

$$e(G_2) \leq (k - 3)|N(K)| + (k - 2)|K| - \binom{k-1}{2} + 2 + d. \quad (\text{a})$$

Let $t = e_G(N(K), K)$. We have $e(G_2) = e(K) + t + e(N(K))$ and

$$2e(K) \geq (2k - 4)|K| - t, \quad (\text{b})$$

and hence

$$e(G_2) \geq (k - 2)|K| + t/2 + d|N(K)|/2. \quad (\text{c})$$

Since $N(x)$ has minimum degree at least $k - 2$, it follows that the subgraph $N(K)$ of $N(x)$ has minimum degree at least $(k - 2) - (d(x) - |N(K)|)$. Thus $d \geq (k - 2) - (d(x) - |N(K)|) \geq |N(K)| - k + 3$. From (a) and (c) we get

$$-t/2 \geq -(k - 3)|N(K)| + d(|N(K)| - 2)/2 + \binom{k-1}{2} - 2 \geq \begin{cases} -8 & \text{if } k = 7 \\ -14 & \text{if } k = 8 \\ -18 & \text{if } k = 9 \end{cases} \quad (\text{d})$$

where the second inequality becomes $\frac{t}{2} \leq 11$ when $|N(K)| = 2k - 6$ and $k = 7, 8$, and the second inequality holds with equality only when $|N(K)| = 10$ and $k = 9$. Since G is not contractible to

K_k , we deduce from (b) and Theorem 1.8, Theorem 1.9 and Theorem 1.10 that $|K| < 8$. The inequalities $e(K) \geq 5|K| - 8$ when $k = 7$, $e(K) \geq 6|K| - 14$ when $k = 8$, and $e(K) \geq 7|K| - 18$ when $k = 9$ imply $|K| \leq 3$. But every vertex of K has degree at least $2k - 4$ and $N(K)$ is a proper subgraph of $N(x)$, and hence $|K| = 3$, $|N(K)| = 2k - 6$ and $\frac{t}{2} = 3(k - 3) \geq 12$ when $k = 7, 8$, and (d) holds with equality for $|N(K)| = 12$ when $k = 9$, contrary to our earlier observation of (d) that $\frac{t}{2} \leq 11$ when $|N(K)| = 2k - 6$ and $k = 7, 8$, and (d) holds with equality only when $|N(K)| = 10$ and $k = 9$. ■

By (1) there is a vertex x of degree $k + 1, k + 2, \dots$, or $2k - 5$ in G . Choose such a vertex x so that $G - N[x]$ has a component K of minimum order. Then choose a vertex $y \in V(K)$ of least degree in G . Thus $k + 1 \leq d_G(y) \leq 2k - 5$ by (1) and (5). Let L be the component of $G - N[y]$ containing x . We claim that $N(L)$ contains all vertices of $N(y)$ that are not adjacent to all other vertices of $N(y)$. Indeed, let $z \in N(y)$ be not adjacent to some vertex of $N(y) - \{z\}$. We may assume that $z \notin N(x)$, for otherwise $z \in N(L)$. Thus $z \in V(K)$, and hence $d_G(z) \geq d_G(y)$ by the choice of y . Thus z has a neighbor $z' \in N[x] \cup K - N[y]$. Then $z' \in V(L)$, for otherwise the component of $G - N[y]$ containing z' would be a proper subgraph of K . Thus $z \in N(L)$. This proves our claim that $N(L)$ contains all vertices z as above, contrary to (3). This contradiction completes the proof of Theorem 1.7. ■

We are now ready to prove Theorem 1.6.

Proof. Let G be a double-critical t -chromatic graph with $t \geq k$. The assertion is trivially true if G is complete. By Theorem 1.2, we may assume that $t \geq 6$. By Proposition 2.1 (a), $\delta(G) \geq k + 1$. By Theorem 1.8, Theorem 1.9 and Theorem 1.10, we have $\delta(G) \leq 2k - 5$ and so $k + 1 \leq \delta(G) \leq 2k - 5$. By Proposition 2.1 (b), every edge of G is contained in at least $k - 2$ triangles. By Proposition 2.1 (c), G is 6-connected and no minimal separating set of G can be partitioned into a clique and an independent set. By Theorem 1.7, $G > K_k$, as desired. ■

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